Electronic Journal of Mathematical Analysis and Applications
Vol. 8(2) July 2020, pp. 231-243.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

AN INFINITE-DIMENSIONAL SUBSPACE OF A NON-NORMABLE AND SEPARABLE FRÉCHET SPACE

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ABSTRACT. In this paper, we proved that if F is a non-normable and separable Fréchet space, then there exists an infinite-dimensional subspace $\mathcal{A} \subset L(F)$ such that any non-zero operator $T \in \mathcal{A}$ is hypercyclic. We considered the existing partial solutions due to Bernal-González [15] and Bès and Conejero [9] to develop our results. An illustrative example is also provided.

1. INTRODUCTION

Hypercyclicity of continuous linear operators on non-normable and separable Fréchet spaces has been considered by several authors like Gethner and Shapiro [19], Godefroy and Shapiro [6] and many others. Ansari [22], Bernal-González [14], and Bonet and Peris [10] independently proved that, every separable infinite-dimensional Fréchet space admits a hypercyclic operator. Rolewicz [21] showed that every scalar multiple μB is hypercyclic on ℓ_2 whenever the scalar μ has modulus strictly larger than 1 and $B : (v_1, v_2, v_3, \cdots) \mapsto (v_2, v_3, v_4, \cdots)$ is the backward shift operator, but Montes [1] showed that no such operators have a hypercyclic subspace. Read [2], and Bernal-González and Montes-Rodríguez [16] constructed the first examples of hypercyclic subspaces.

González et al. [17] proved that if an operator T acting on a Banach space B satisfies that $T \oplus T$ is hypercyclic on $B \times B$, then T has a hypercyclic subspace if and only if there exists a closed, infinite-dimensional subspace B_0 of B and integers $1 < k_1 < k_2 < \cdots$ so that,

$$T^{k_n} v \longrightarrow 0 \; as \; n \longrightarrow \infty \text{ for every } v \in B_0 \tag{1}$$

and moreover, if and only if the essential spectrum of T meets the closed unit disk. Also, condition (1) above is used to prove the existence of a hypercyclic subspace on Fréchet space F with a continuous norm. For example, Bernal-González [15] and Petersson [8] independently used this fact to prove that every separable infinite-dimensional Fréchet space with a continuous norm admits a hypercyclic subspace. Further, Bonet et al. [11] proved that in general, condition (1) above is not sufficient

2010 *Mathematics Subject Classification.* 47A16, 47A15.
Key words and phrases. Non-normable Fréchet space, Hypercyclic operator, Hypercyclic subspace, Hypercyclic manifold.
Submitted Jan. 4, 2019. Revised June 18, 2019.

in the case of Fréchet spaces without a continuous norm, that is, the operator $T : (v_j)_{j\in\mathbb{Z}} \mapsto (2v_{j+1})_{j\in\mathbb{Z}}$ acting on $F = \{(v_j)_{j\in\mathbb{Z}} \in \mathbb{K}^{\mathbb{Z}} : (v_j)_{j=1}^{\infty} \in \ell_2\}$ satisfies condition (1) and $T \oplus T$ is hypercyclic and eventually T does not have a hypercyclic subspace.

Furthermore, Bès and Conejero [9] proved that any countable family of operators of the form $P(B)$, where P is a non-constant polynomial and B is the backward shift operator on ω, the countably infinite product of lines, has a common hypercyclic subspace.

Following is the definition of hypercyclicity:

Definition 1.1. [10] *An operator T on a locally convex space F is called hypercyclic if $Orb(T, v) := \{v, Tv, T^2v, \cdots\}$ is dense in F for some $v \in F$, that is,*

$$\overline{Orb(T, v)} = \overline{\{v, Tv, T^2v, \cdots\}} = F.$$

In this case, v is a hypercyclic vector for T.

In this work, we present the results for an infinite-dimensional subspace of a non-normable and separable Fréchet space and throughout this work an operator means a continuous linear map and $L(F)$ is the space of all operators $T : F \longrightarrow F$. The strong operator topology (SOT) in $L(F)$ is the one where the convergence is defined as pointwise convergence at every $v \in F$.

The main goal of this paper is to investigate the following question:

Question 1.1. [23] *If F is a non-normable and separable Fréchet space, is there an infinite-dimensional subspace $\mathcal{A} \subset L(F)$ such that any non-zero operator $T \in \mathcal{A}$ is hypercyclic?*

2. Preliminaries

To establish the main results for this paper, we will require the following definitions, lemmas and theorems:

Definition 2.1. [4] *Let F be a topological vector space. $T \in L(F)$ is said to satisfy the Hypercyclicity criterion if there exists an increasing sequence of integers (n_k), two dense sets $A, B \subset F$ and a sequence of maps $S_{n_k} : B \longrightarrow F$ such that*

(i) $T^{n_k}(x) \longrightarrow 0$ *for any* $x \in A$*;*
(ii) $S_{n_k}(y) \longrightarrow 0$ *for any* $y \in B$*;*
(iii) $T^{n_k}S_{n_k}(y) \longrightarrow y$ *for each* $y \in B$*.*

Definition 2.2. [18] *Let A be a set of points in the plane. The convex hull of A is the smallest convex polygon that contains all the points of A. That is, for any subset of the plane (set of points, rectangle, simple polygon), its convex hull is the smallest convex set that contains that subset.*

Definition 2.3. [5] *A topological vector space F is called normable if its topology can be defined by a norm, that is, if there is a norm $||\cdot||$ on F such that the balls*

$$B_\varepsilon = \{v \in F : ||v|| \le \varepsilon\},\ \varepsilon > 0,$$

form a basis of neighborhoods of the origin.

Remark 2.1. [5] *Finite-dimensional Hausdorff spaces are normable while in general, not all infinite-dimensional metrizable topological vector spaces are normable.*

Definition 2.4. [9] *Let T be a continuous linear operator acting on a Fréchet space F. A hypercyclic manifold for T is a dense, invariant subspace of F consisting entirely, except for the origin, of hypercyclic vectors for T. A hypercyclic subspace for T is a closed, infinite-dimensional subspace of F consisting entirely, except for the origin, of hypercyclic vectors for T.*

Lemma 2.1. [20] *Let F be one of the sequence spaces l_p $(1 \leq p < \infty)$ or c_0, and let $(w_k)_{k\geq 0}$ be any bounded sequence of positive scalars. Consider the operator S defined on F by*

$$S(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \cdots) = (w_0\alpha_1, w_1\alpha_3, w_2\alpha_5, \cdots).$$

Then the operator $I + S$ is hypercyclic on F. In fact, it satisfies Hypercyclicity criterion.

Lemma 2.2. [15] *Assume that E, F are complete and metrizable topological vector spaces, $T \in L(E)$, $\tilde{T} \in L(F)$, $Q \in L(F, E)$ and Q has dense range and that $Q\tilde{T} = TQ$. If $\tilde{T}$ satisfies the Hypercyclicity criterion, then T does also.*

Lemma 2.3. [15] *Assume that $(n_k) \in \mathbb{N}$, E and F is Fréchet space and $S_m \subset L(E, F)$. Suppose that following properties are satisfied:*

(a) *The space E admits a norm which is continuous,*
(b) *The sequence S_m satisfies the Hypercyclicity criterion with respect to (n_k),*
(c) *There exists a closed infinite-dimensional subspace $E_0 \subset E$ such that for every $y \in E_0$, the sequence $(S_{n_k} y)$ converges in F.*

Then (S_{n_k}) admits a hypercyclic subspace.

Remark 2.2. *The above conclusion says in particular that (S_n) admits a hypercyclic subspace.*

Lemma 2.4. [15] *Let F be an infinite-dimensional locally convex space, and let $T \in L(F)$ be fixed. Then following statements are equivalent:*

(a) *The set of conjugates $\{STS^{-1} : S \text{ invertible}\}$ of T is strong operator topology (SOT)-dense in $L(F)$.*
(b) *For all $k \in \mathbb{N}$, there exist $h_1, \cdots, h_k$ in F so that the set $\{h_1, \cdots, h_k, Th_1, \cdots, Th_k\}$ is linearly independent.*

Theorem 2.1. [15] *Suppose that F is a separable infinite-dimensional Fréchet space admitting a continuous norm. Then F supports an operator which possesses a hypercyclic subspace. Even more, the family of such operators is SOT-dense in $L(F)$.*

Theorem 2.2. [13] *Let F be a separable Fréchet space with a continuous norm, and let T be an operator on F. Suppose that there exists an increasing sequence $(k_n)_n$ of positive integers such that*

(a) *T satisfies the Hypercyclicity Criterion for $(k_n)_n$,*
(b) *there exists an infinite-dimensional closed subspace N_0 of F such that $T^{k_n}v \longrightarrow 0$ for all $v \in N_0$.*

Then T has a hypercyclic subspace.

Theorem 2.3. [9] *Let $(P_n)_{n=1}^{\infty}$ be any sequence of non-constant polynomials and let B be the backward shift operator on ω. Then the operators $P_n(B)(n \in \mathbb{N})$ have a common hypercyclic subspace. That is, there exists a closed infinite dimensional subspace $\mathcal{A}$ of ω satisfying*

$$\{v, P_n(B)v, P_n^2(B)v, \cdots\}$$

is dense in ω for each $0 \neq v \in \mathcal{A}$ and each $n \in \mathbb{N}$.

Bès and Conejero [9] showed two lemmas before proving Theorem 2.3 above. Let Π_k denote the standard projection of ω into $\mathbb{K}^k$, for each $k \in \mathbb{N}$; that is, $\Pi_k x = (x_1, \cdots x_k)$ for each $x = (x_i)_{i=1}^{\infty}$ in ω.

Lemma 2.5. [9] *Let $T = P(B)$, where B is the backward shift on ω and $P(t) = a_1 + a_2 t + \cdots + a_{d+1}t^d$ is any polynomial of degree $d \geq 1$. Then for each $l, k \in \mathbb{N}$, $(y_1, y_2, \cdots y_l) \in \mathbb{K}^l$ and $(x_1, x_2, \cdots, x_{kd}) \in \mathbb{K}^{kd}$, there exists a unique $(z_1, z_2, \cdots, z_l) \in \mathbb{K}^l$ so that*

$$\Pi_l T^k(x_1, x_2, \cdots x_{kd}, z_1, z_2, \cdots, z_l), (h_1, h_2, \cdots) = (y_1, y_2, \cdots y_l)$$

for each $h_1, h_2, \cdots$ in $\mathbb{K}$.

Lemma 2.6. [9] *Let $[f_{i,j}] \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}}$ be an infinite matrix with coefficients in $\mathbb{K}$ and no row of zeros. For each row $f_k = (f_{k,1}, f_{k,2}, \cdots)$, let $j_k := min\{j \in \mathbb{N} : f_{k,j} \neq 0\}$. Suppose further that $(j_n)_{n=1}^{\infty}$ is strictly increasing, then*

(a) *$\{f_1, f_2, \cdots\}$ is linearly independent, and*
(b) *$\overline{span\{f_1, f_2, \cdots\}}^{\omega} = \{\sum_{k=1}^{\infty} \alpha_k f_k : (\alpha_k)_{k=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}\}$.*

3. Main Results

Now, we will respond to Question 1.1 by considering two cases; the case of a non-normable and separable Fréchet space which admits a norm which is continuous and the case of a non-normable and separable Fréchet space which admits a norm which is not continuous.

3.1. The case of a non-normable and separable Fréchet space which admits a norm which is continuous.

We extend Theorem 2.1 from separable infinite-dimensional Fréchet space to non-normable and separable Fréchet space.

Theorem 3.1. *Suppose that F is a non-normable and separable infinite-dimensional Fréchet space admitting a continuous norm. Then F supports an operator which possesses a hypercyclic subspace. Even more, the family of such operators is SOT-dense in $L(F)$.*

Proof. The second statement of the theorem is acquired from the first together with Lemma 2.4. Since the family under consideration is invariant under conjugation and F is infinite-dimensional, each hyperyclic operator has at least one dense orbit, say $\{v_0, Tv_0, T^2v_0, \cdots\}$. Hence it is sufficient to show that the operator $T \in L(F)$ possesses a hypercyclic subspace.

It is known that the countable product of lines $\omega := \mathbb{K}^{\mathbb{N}}$ endowed with the product topology is a Fréchet space which admits a norm which is not continuous, so $F \neq \omega$. Then by Bonet and Peris [10] (Lemma 2 and Theorem 1), there are sequences $(x_k)_{k\geq 0} \subset F$ and $(f_k)_{k\geq 0} \subset F'$ fulfilling the following conditions:

(i) (x_k) converge to $0 \in F$, and the closed absolutely convex hull C of (x_k) satisfies that, there is a Banach space F_C which is dense in F.
(ii) (f_k) is F-equicontinuous in F'.
(iii) $f_m(x_k) = 0$ if $m \neq k$ and $\{f_k(x_k) : k \geq 0\} \subset (0, 1)$.

Now, consider the operator T on F defined as $T := I + S$, where

$$Sx := \sum_{k=0}^{\infty} \frac{f_{2k+1}(x)}{2^k} x_k,\ x \in F.$$

In the same way as in the proof of Lemma 3 by Bonet and Peris [10], we obtain

$$C = \left\{ \sum_{k=0}^{\infty} \alpha_k x_k \ : \ \sum_{k=0}^{\infty} |\alpha_k| \leq 1 \right\} \tag{2}$$

and the mapping

$$Q : \alpha = (\alpha_k)_{k\geq 0} \in l_1 \mapsto \sum_{k=0}^{\infty} \alpha_k x_k \in F_C$$

is linear, continuous and surjective. By Lemma 2.1, the operator

$$\tilde{T} : (\alpha_k)_{k\geq 0} \in l_1 \mapsto \left(\alpha_0 + f_0(x_0)\alpha_1, \alpha_1 + \frac{f_1(x_1)}{2}\alpha_3, \alpha_2 + \frac{f_2(x_2)}{2^2}\alpha_5, \cdots \right) \in l_1$$

satisfies the Hypercyclicity criterion since the weights $w_k = \frac{f_k(x_k)}{2^k}$ are positive and form a bounded sequence.

We need to show that T also satisfies the Hypercyclicity criterion. If we consider Q as a mapping $Q : l_1 \longrightarrow F$, then Q is also continuous. Clearly, such a mapping is the composition of $Q : l_1 \longrightarrow F_C$ with the canonical inclusion $F_C \longrightarrow F$. If

$$\{u_n := \sum_{k=0}^{\infty} \alpha_{k_n} x_k\},\ n \in \mathbb{N} \subset F_C$$

is a sequence tending to zero in the topology of F_C, then it also tends to zero in the topology of F, this implies that the inclusion is linear and continuous because by Equation (2) we have

$$\lambda C = \left\{ \sum_{k=0}^{\infty} \alpha_k x_k \ : \ \sum_{k=0}^{\infty} |\alpha_k| \leq \lambda \right\}$$

for all $\lambda > 0$.

By condition (iii) above, the series expansion

$$x = \sum_{k=0}^{\infty} \alpha_k x_k$$

of each $x \in F_C$ in terms of some sequence $(\alpha_k) \subset \mathbb{K}$ is unique. So if $\lambda > 0$ and $x \in \lambda C$ we get,

$$\sum_{k=0}^{\infty} |\alpha_k| \leq \lambda, \text{ thus } \sum_{k=0}^{\infty} |\alpha_k| \leq p_B(x).$$

But

$$\lim_{n\to\infty} p_B(u_n) = 0 \text{ whereas } \lim_{n\to\infty} \sum_{k=0}^{\infty} |\alpha_{k_n}| = 0.$$

Further, F is locally convex, so in order to show that (u_n) tends to zero in F it is sufficient to show that $u_n \longrightarrow 0$ in the Mackey topology, that is,

$$\lim_{n\to\infty} \sup_{\varphi\in A} |\varphi(u_n)| = 0$$

for every equicontinuous subset $A \subset F'$ (for example, see Proposition 7 by Horváth [12]).

Since x_k is bounded in F, it is true that if A is equicontinuous, then there is a constant $K \in (0, \infty)$ such that

$$\sup_{\varphi\in A,\ k\geq 0} |\varphi(u_n)| \leq K.$$

Therefore,

$$\begin{aligned}\sup_{\varphi\in A} |\varphi(u_n)| =& \sup_{\varphi\in A} \left|\varphi\left(\sum_{k=0}^{\infty} \alpha_{k_n} x_k\right)\right| \\ \leq& \sum_{k=0}^{\infty} |\alpha_{k_n}| \sup_{\varphi\in A,\ k\geq 0} |\varphi(x_k)| \\ \leq& K \sum_{k=0}^{\infty} |\alpha_{k_n}| \to 0 \text{ as } n \to \infty,\end{aligned}$$

from which the left-hand side approaches to zero as $k \to \infty$ as desired.

Hence, $Q : l_1 \longrightarrow F$ is linear and continuous and also has a dense range due to $Q(l_1) = F_C$ and condition (i) and the fact that $TQ = Q\tilde{T}$ on l_1. Recalling that $T \in L(F), \tilde{T} \in L(l_1)$ and $\tilde{T}$ satisfies the Hypercyclicity criterion, and setting $F := l_1$ in Lemma 2.2, we conclude that T satisfies the Hypercyclicity criterion. Notice that the restriction of the operator $\tilde{T}$ to the subspace

$$G := \{\alpha \in l_1 : \alpha_{2k+1} = 0,\ \forall\ k \geq 0\},$$

is the identity operator. Consider the linear manifold

$$G_0 := Q(G) \subset F_C \subset F.$$

Since $TQ = Q\tilde{T}$ on l_1, the observation that the restriction of the operator $\tilde{T}$ to G is the identity yields that T is the identity on G_0, that is, $Tx = x$ for all $x \in G_0$, and by continuity we get $Tx = x$ for all $x \in F_0$ where

$$F_0 := \text{closure}_F(G_0). \tag{3}$$

From condition (iii) above, we can easily deduce that the sequence x_k is linearly independent. Thus, $\text{span}(\{x_{2k} : k \geq 0\})$ is infinite-dimensional, hence by Equation (3) and by the inclusions

$$\text{span}(\{x_{2k} : k \geq 0\}) \subset G_0 \subset F_0,$$

we obtain that F_0 is a closed infinite-dimensional subspace of F.

Lastly, if $x \in F_0$ then $Tx = x$ so $S_k x = x$ for every $k \in \mathbb{N}$ and every $x \in F_0$, where we have set $S_k := T^k$ for every $k \in \mathbb{N}$ and hence, $\lim_{k\to\infty} S_k x = x$ exists for all $x \in F_0$. Therefore by the application of Lemma 2.3 with $F := E$, we conclude that $(S_{m_n} x)$ converges in F for every $x \in F_0$, where $m_n \subset \mathbb{N}$ is a sequence with respect to which T or the sequence S_k satisfies the Hypercyclicity criterion. □

3.2. **The case of a non-normable and separable Fréchet space which admits a norm which is not continuous.** We consider a non-normable and separable Fréchet space $F = \omega := \mathbb{K}^{\mathbb{N}}$ that is, the countable product of lines given the product topology. Even though $F = \omega$ does not have a dense subspace with a continuous norm, we still can use Theorem 2.3 by Bès and Conejero [9] to show that it supports operators with a hypercyclic subspace.

Our main result in this subsection is stated in Theorem 3.2 and we will require following two lemmas:

Let Π_k denote the standard projection of $F = \omega$ into $\mathbb{K}^k$, for each $k \in \mathbb{N}$; that is, $\Pi_k v = (v_1, \cdots v_k)$ for each $v = (v_i)_{i=1}^{\infty}$ in $F = \omega$.

Lemma 3.1. *Let $T = P(B)$, where B is the backward shift on $F = \omega$ and $P(t) = a_1 + a_2 t + \cdots + a_{d+1} t^d$ is any polynomial of degree $d \geq 1$. Then for each $l, k \in \mathbb{N}$, $(u_1, u_2, \cdots u_l) \in \mathbb{K}^l$ and $(v_1, v_2, \cdots, v_{kd}) \in \mathbb{K}^{kd}$, there exists a unique $(w_1, w_2, \cdots, w_l) \in \mathbb{K}^l$ so that*

$$\Pi_l T^k (v_1, v_2, \cdots v_{kd}, w_1, w_2, \cdots, w_l, h_1, h_2, \cdots) = (u_1, u_2, \cdots u_l)$$

for each $h_1, h_2, \cdots$ in $\mathbb{K}$.

Proof. For every $v = (v_i)_{i=1}^{\infty}$ in $F = \omega$, we have

$$Tv = \left(a_1 v_j + a_2 v_{j+1} + \cdots + a_d v_{j+d-1} + a_{d+1} v_{j+d} \right)_{j=1}^{\infty},$$

and generally, for every $k \in \mathbb{N}$ the kth iterate of T is of the form

$$T^k v = \left(\varphi_{k,j}(v_1, v_2, \cdots, v_{j+kd-1}) + (a_{d+1})^k v_{j+kd} \right)_{j=1}^{\infty},$$

for some linear functions

$$\varphi_{k,j} : \mathbb{K}^{kd+j-1} \to \mathbb{K},\ j \in \mathbb{N}$$

that are independent of v. Therefore, the lemma follows since $a_{d+1} \neq 0$.

Remark 3.1. *Notice that in Lemma 3.1 we have,*

(i) *If $u_1 = \cdots = u_l = v_1 = \cdots = v_{kd} = 0$, then $w_1 = \cdots = w_l = 0$.*
(ii) *If $l = 1$ and $u_1 \neq 0$ and $v_1 = \cdots = v_{kd} = 0$, then $w_1 \neq 0$.*

□

Lemma 3.2. *Let $[f_{i,j}] \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}}$ be an infinite matrix with coefficients in $\mathbb{K}$ and no row of zeros. For each row $f_k = (f_{k,1}, f_{k,2}, \cdots)$, let $j_k := min\{j \in \mathbb{N} : f_{k,j} \neq 0\}$. Moreover, if $(j_n)_{n=1}^{\infty}$ is strictly increasing, then*

(a) *$\{f_1, f_2, \cdots\}$ is linearly independent, and*

(b) $\overline{span\{f_1, f_2, \cdots\}}^{F=\omega} = \{\sum_{k=1}^{\infty} \alpha_k f_k : (\alpha_k)_{k=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}\}$.

Proof. For each $s \in \mathbb{N}$ we have $f_{s,j_s} \neq 0$ and $f_{k,j} = 0$ for every $(k, j) \in (s, \infty) \times [1, j_s]$, since $(j_n)_{n=1}^{\infty}$ is strictly increasing. Hence (a) follows and $\sum_{k=1}^{\infty} \alpha_k f_k$ converges in $F = \omega$ for any $(\alpha_k)_{k=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}$. Let

$$g \in \overline{span\{f_1, f_2, \cdots\}}^{F=\omega},$$

then there exist integers $1 < r_1 < r_2 < \cdots$ and sequences

$$(\alpha_{k,1})_{k=1}^{\infty}, (\alpha_{k,2})_{k=1}^{\infty}, \cdots \in \mathbb{K}$$

so that

$$h_k := (\alpha_{k,1} f_1 + \alpha_{k,2} f_2 + \cdots + \alpha_{k,r_k} f_{r_k}) \to g, \text{ as } k \to \infty. \tag{4}$$

We prove that there exists a sequence $(\alpha_s)_{s=1}^{\infty} \in \mathbb{K}$ so that

$$\Pi_{j_s} = (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_s f_s) = \Pi_{j_s}(g), \ s \in \mathbb{N}. \tag{5}$$

Now,

$$\alpha_{k,1}(f_{1,1}, f_{1,2}, \cdots, f_{1,j_1}) = \Pi_{j_1}(h_k)$$

and so by Equation (4), $\alpha_{k,1} \to \alpha_1$ as $k \to \infty$ and

$$\Pi_{j_1}(g) = \Pi_{j_1}(\alpha_1 f_1), \text{ where } \alpha_1 = g_{j_1}/f_{1,j_1}.$$

Inductively, suppose that we found $\alpha_i \in \mathbb{K}$ where $(1 \leq i \leq s-1)$ so that

$$\alpha_{k,i} \to \alpha_i \text{ as } k \to \infty \text{ and } \Pi_{j_i}(g) = \Pi_{j_i}(\alpha_1 f_1 + \cdots + \alpha_i f_i) \tag{6}$$

for every $(1 \leq i \leq s-1)$.

On the other hand, since $(j_n)_{n=1}^{\infty}$ is strictly increasing, then

$$\Pi_{j_s}(\alpha_{k,1} f_1 + \cdots + \alpha_{k,s} f_s) = \Pi_{j_s}(h_k)$$

and so by Equations (6) and (4) we have $\alpha_{k,s} \to \alpha_s$ as $k \to \infty$ and

$$\Pi_{j_s}(g) = \Pi_{j_s}(\alpha_1 f_1 + \cdots + \alpha_s f_s),$$
$$\text{where } \alpha_s = (g_{j_s}(\alpha_1 f_{1,j_s} + \cdots + \alpha_{s-1} f_{s-1,j_s}))/f_{s,j_s},$$

hence Equation (5) follows.

□

Theorem 3.2. *Let $(P_n)_{n=1}^{\infty}$ be any sequence of non-constant polynomials and let B be the backward shift operator on $F = \omega$. Then the operators $P_n(B)(n \in \mathbb{N})$ have a common hypercyclic subspace. That is, there exists a closed infinite dimensional subspace $\mathcal{A}$ of $F = \omega$ satisfying that*

$$\{v, P_n(B)v, P_n^2(B)v, \cdots\}$$

is dense in $F = \omega$ for each $0 \neq v \in \mathcal{A}$ and each $n \in \mathbb{N}$.

Proof. Let $\{r_l : l \in \mathbb{N}\}$ be a countable dense set in $F = \omega$ so that every $r_l = (r_{l,j})_{j=1}^{\infty}$ satisfies that $r_{l,j} \neq 0$ if and only if $1 \leq j \leq l$. For every $n \in \mathbb{N}$, let $T_n := P_n(B)$ and $d_n :=$ degree (P_n).

In proving Theorem 3.2 above, we will use the following statement:

Step 3.1. *There exists an infinite, upper triangular matrix $H = [f_{i,j}] \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}}$ satisfying*

1. *no row $f_k = (f_{k,1}, f_{k,2}, \cdots)$ is zero.*
2. *the sequence $(j_k)_{k=1}^{\infty}$ given by $j_k := min\{f_{k,j} \neq 0 : \ j \in \mathbb{N}\}$ is strictly increasing.*
3. *for every $(n, i, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with $n < i + l$, there exists a positive integer $m_{n,i,l}$ so that*

$$\mathit{\Pi}_l T_n^{m_{n,i,l}} f_k = \begin{cases} (r_{l,1}, r_{l,2}, \cdots, r_{l,l}) & if \quad k = i, \\ (0, 0, \cdots, 0) & if \quad k \neq i. \end{cases}$$

Suppose the statement holds, then we show that

$$S := \overline{span\{f_1, f_2, \cdots\}^{F=\omega}}$$

is a hypercyclic subspace for every T_n where $n \in \mathbb{N}$. By conditions (1), (2), and Lemma 3.2(a), the closed subspace S is infinite-dimensional. Let $0 \neq f \in S$, we now prove that f is hypercyclic for T_n, $n \in \mathbb{N}$. By Lemma 3.2, f can written as

$$f = \sum_{k=1}^{\infty} \alpha_k f_k$$

for some sequence of scalars $(\alpha_k)_{k=1}^{\infty}$. If necessary, multiplying f by a non-zero scalar, we may assume without loss of generality that $\alpha_i = 1$ for some $i \in \mathbb{N}$. But by condition (3), for every $l > \max\{n - i, 1\}$, one has

$$\mathit{\Pi}_l T_n^{m_{n,i,l}} f = \sum_{k=1}^{\infty} \alpha_k \mathit{\Pi}_l T_n^{m_{n,i,l}} f_k = \mathit{\Pi}_l T_n^{m_{n,i,l}} f_i = (r_{l,1}, r_{l,2}. \cdots, r_{l,l}).$$

Hence, it follows that f is hypercyclic for $T_n,\ n \in \mathbb{N}$.

We now proceed with the proof of the theorem by proving the statement in Step (3.1) above, as follows:

Let $N_{0,0} := 1$. Inductively, for every $M \in \mathbb{N}$ define

$$\mathit{\Pi}_l T_n^{m_{n,i,l}} f_k = \begin{cases} N_M := d_M N_{(M-1),(M-1)^2}, \\ N_{M,i} := 2^{M+i} N_M \ (1 \leq i \leq N^2), \\ N_{(M-1),(M-1)^2+1} := N_{M,1}. \end{cases} \tag{7}$$

Also, for every $(n, i, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with $1 \leq n \leq i + l - 1$ let

$$m_{n,i,l} := \frac{N_{(i+l-1),((n-1)(i+l-1)+i)}}{d_n}.$$

Finally, let $f_{k,j} = 0$ for every $(k, j) \in \mathbb{N} \times [1, N_{1,1}]$. Then define the matrix $H = [f_{k,j}] \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}}$ of the statement in Step 3.1 above inductively, where by in each inductive step M, we let $f_{k,j} = 0$ for all $(k, j) \in \mathbb{N} \times (N_{M,1}, N_{M+1,1}]$.

Step 3.2. *When $M = 1$.*

We define $f_{k,j}$ for all $(k,j) \in \mathbb{N} \times (N_{1,1}, N_{2,1}]$ so that

$$\Pi_1 T_1^{m_{1,1,1}} g_k = \begin{cases} r_{1,1} & if \quad k = 1, \\ 0 & if \quad k \neq 1, \end{cases} \tag{8}$$

for any $g_k \in F = \omega$ of the form $g_k = (f_{k,1}, f_{k,2}, \cdots, f_{k,N_{2,1}}, *, *, \cdots)$. Letting $l = 1, m = m_{1,1,1}, T = T_1, d = d_1, u_1 = r_{1,1}$ and $v_j = f_{1,j}$ $(1 \leq j \leq N_{1,1})$, by Lemma 3.1 there exists a unique $w \in \mathbb{K}$ so that

$$\Pi_1 T_1^{m_{1,1,1}} (f_{1,1}, f_{1,2}, \cdots, f_{1,N_{1,1}}, w, *, *, \cdots) = r_{1,1}.$$

If we define $f_{1,N_{1,1}+1} := w$, and $f_{k,j} = 0$ for every $(1, N_{1,1}+1) \neq (k,j) \in \mathbb{N} \times (N_{1,1}, N_{2,1}]$, then Equation (8) is satisfied. By Remark 3.1(ii), we notice that $f_{1,N_{1,1}+1} = w \neq 0$.

Step 3.3. *When $M \geq 2$.*

We divide this step into M^2 substeps for each $(n,i) \in [1,M] \times [1,M]$. Let us start with Substep (3.2.1) $M.1.1$ below and follow with the lexicographic order given by the relation $(n', i') < (n,i)$ if and only if either $n' < n$ or both $n' = n$ and $i' < i$.

In each substep $M.n.i$ we define the coordinates $f_{k,j}$ for all indexes

$$(k,j) \in \mathbb{N} \times (N_{M,(n-1)M+i}, N_{M,(n-1)M+i+1}],$$

so that

$$\Pi_l T_n^{m_{n,i,l}} g_k = \begin{cases} (r_{l,1}, \cdots, r_{l,l}) & if \quad k = i \\ (0, \cdots, 0) & if \quad k \neq i \end{cases} \tag{9}$$

for any g_k of the form $g_k = (f_{k,1}, \cdots, f_{k,N_{M,(n-1)M+i+1}}, *, *, \cdots)$ and $l = M + 1 - i$. By Equation (7), we notice that $N_{M,(n-1)M+i} \leq N_{M,(n-1)M+i+1}$ whenever $(n,i) \in [1,M] \times [1,M]$.

Substep 3.2.1. $M.1.1$.

Applying Lemma 3.1 M-times and considering for every $1 \leq k \leq M : l = M, m = m_{1,1,M}, T = T_1, d = d_1, v_1^{(k)} = f_{k,j}$ $(1 \leq j \leq N_{M,1})$ and $(u_1^{(k)}, \cdots, u_M^{(k)}) = (r_{M,1}, \cdots, r_{M,M})$ if $k = 1$ and $(u_1^{(k)}, \cdots, u_M^{(k)}) = (0, \cdots, 0) \in \mathbb{K}^M$ if $k \neq 1$, we get

$$(w_1^{(k)}, w_2^{(k)}, \cdots, w_M^{(k)}) \in \mathbb{K}^M \ (1 \leq k \leq M)$$

so that

$$\Pi_M T_1^{m_{1,1,M}} g_k = \begin{cases} (r_{M,1}, \cdots, r_{M,M}) & if \quad k = 1, \\ (0, \cdots, 0) & if \quad k \neq 1, \end{cases} \tag{10}$$

for any g_k of the form

$$g_k = (f_{k,1}, \cdots, f_k, N_{M,1}, w_1^{(k)}, \cdots, w_M^{(k)}), *, *, \cdots).$$

Hence Equation (9) is satisfied for $(n,i) = (1,1)$ if we define

$$(f_k, N_{M,1} + 1, \cdots, f_k, N_{M,1} + M) = (w_1^{(k)}, \cdots, w_M^{(k)}) \ (1 \leq k \leq M)$$

and $f_{k,j} = 0$ for every (k,j) in either $(\mathbb{N} \backslash \{1, \cdots, M\}) \times (N_{M,1}, N_{M,2}]$ or in $\mathbb{N} \times (N_{M,1} + N + 1, N_{M,2}]$.

Substep 3.2.2. *M.n.i.*

We have already defined $f_{k,j}$ for every $(k,j) \in \mathbb{N} \times [1, N_{M,(n-1)M+i}]$ so that Equation (9) holds for every $(1,1) \leq (n', i') < (n, i)$. That is,

$$\mathit{\Pi}_l T_{n'}^{m_{n',i',l}} g_k = \begin{cases} (r_{l,1}, \cdots, r_{l,l}) & if \quad k = i' \\ (0, \cdots, 0) & if \quad k \neq i' \end{cases} \tag{11}$$

for any $g_k \in F = \omega$ of the form $g_k = (f_{k,1}, \cdots, f_{k,N_{M,(n'-1)M+i'+1}}, *, *, \cdots)$ and $l = M + 1 - i'$.

Applying Lemma 3.1 M-times and considering for every $1 \leq k \leq M : l = M + 1 - i$,

$$m = m_{n,i,l}, T = T_n, d = d_n, v_j^{(k)} = f_{k,j} \ (1 \leq j \leq N_{M,(n-1)M+i}),$$

$$\text{and } (u_1^{(k)}, \cdots, u_l^{(k)}) = (r_{l,1}, \cdots, r_{l,l})$$

if $k = i$ and $(u_1^{(k)}, \cdots, u_l^{(k)}) = (0, \cdots, 0) \in \mathbb{K}^l$ if $k \neq i$, to obtain $(w_1^{(k)}, w_2^{(k)}, \cdots, w_l^{(k)}) \in \mathbb{K}^l$, $(1 \leq k \leq M)$ so that

$$\mathit{\Pi}_l T_n^{m_{n,i,l}} g_k = \begin{cases} (r_{l,1}, \cdots, r_{l,l}) & if \quad k = i, \\ (0, \cdots, 0) & if \quad k \neq i, \end{cases} \tag{12}$$

for any $g_k \in F = \omega$ of the form

$$g_k = (f_{k,1}, \cdots, f_k, N_{M,(n-1)M+i}, w_1^{(k)}, \cdots, w_l^{(k)}), *, *, \cdots)$$

and $l = M + 1 - i$. So Equation (9) is satisfied if we define

$$f_k, N_{M,(n-1)M+i} + s = w_s^{(k)}$$

when $(k, s) \in [1, M] \times [1, l]$, and $f_{k,j=0}$ for all indexes (k, j) in either

$$(\mathbb{N} \backslash \{1, \cdots, M\}) \times (N_{M,(n-1)M+i}, N_{M,(n-1)M+i+1}]$$
$$\text{or } \{1, \cdots, M\} \times (N_{M,(n-1)M+i} + l, N_{M,(n-1)M+i+1}].$$

Now, we have defined completely the matrix $H = [f_{k,j}] \in \mathbb{K}^{\mathbb{N} \times \mathbb{N}}$. We observe that, for every $M \in \mathbb{N}$, $f_{M,j} = 0$ for $1 \leq j \leq N_{M,M}$ and by Remark 3.1(ii), $f_{M,N_{M,M}+1} \neq 0$. Thus,

$$j_M = \min\{f_{M,j} \neq 0 : j \in \mathbb{N}\} = N_{M,M} + 1,$$

and hence conditions (1) and (2) of the Step 3.1 above hold.

Finally, given any $(n, i, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with $n < i + l$, our definitions on Substep (3.2.2) M, n, i of step $M = i + l - 1$ given by Equation (9) ensure that

$$\mathit{\Pi}_l T_n^{m_{n,i,l}} g_k = \begin{cases} (r_{l,1}, \cdots, r_{l,l}) & if \quad k = i, \\ (0, \cdots, 0) & if \quad k \neq i. \end{cases}$$

Therefore, condition (3) of the Step 3.1 above holds.

□

We now present an illustrative example to support the results above. We use Definition 2.1 and Theorem 2.2 above and apply the results by MacLane [7], and Grosse-Erdmann and Manguillot [13] to construct this example, as follows:

Example 3.1. *Let F be a separable Fréchet space that is isomorphic to $H(\mathbb{C})$, the space of all entire functions which has a continuous norm, and let $D : f \mapsto f'$ be the derivative operator on F. Then D has a hypercyclic subspace.*

Proof. The derivative operator $D : f \mapsto f'$ satisfies the Hypercyclicity Criterion for the full sequence since it is hypercyclic (see Example 1.8 by Bayart and Matheron [4]). So condition (a) of the Theorem 2.2 above is satisfied. Now, it remains to show the existence of an infinite-dimensional closed subspace N_0 of $F = H(\mathbb{C})$ on which suitable powers of D approaches to 0.

By starting, we note that for any $k \geq 1$ there is some $C_k > 0$ such that for all $v \geq C_k$,

$$v^k \leq 2^v. \tag{13}$$

Next we choose a strictly increasing sequence of positive integers $(k_n)_n$ with $k_1 \geq 1$ such that

$$k_{n+1} \geq C_{k_n} \text{ for all } n \geq 1.$$

For $i \geq n+1$ we have that $k_i \geq k_{n+1} \geq C_{k_n}$ and hence, by Equation (13) above

$$k_i^{k_n} \leq 2^{k_i} \text{ for } i \geq n+1. \tag{14}$$

Now, let N_0 be the closed subspace of $F = H(\mathbb{C})$ of all entire functions f of the form

$$f(w) = \sum_{n=1}^{\infty} a_n w^{k_n - 1}.$$

We need to show that

$$D^{k_n} f \longrightarrow 0 \text{ in } F = H(\mathbb{C}) \text{ as } n \longrightarrow \infty.$$

Let $R \geq 1$. Then we have,

$$\begin{aligned}
\sup_{|w|\leq R}\left|D^{k_n} f(w)\right| &= \sup_{|w|\leq R}\left|\sum_{i=n+1}^{\infty} a_i D^{k_n} w^{k_i-1}\right| \\
&\leq \sum_{i=n+1}^{\infty} |a_i|(k_i-1)\cdots(k_i-k_n)R^{k_i-k_n-1} \\
&\leq \sum_{i=n+1}^{\infty} |a_i| k_i^{k_n} R^{k_i} \\
&\leq \sum_{i=n+1}^{\infty} |a_i|(2R)^{k_i} \longrightarrow 0 \text{ as } n \longrightarrow \infty,
\end{aligned}$$

where in the last inequality we have used Equation (14). Thus, condition (b) of the Theorem 2.2 above holds. □

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